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SOME RESULTS ON GRADED *N*-PRIME SUBMODULES

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Abstract

In this paper, we consider graded *N*-prime submodules as introduced by Sanh, and investigate their properties besides characterizations. For example, we prove that (i) if X is a fully invariant graded submodule of M, then the residual ideal of X by M is a graded ideal of S, and (ii) if M is a graded quasi-projective module, X is a graded N-prime submodule of M and $Y \subset X$ is a fully invariant graded submodule of M, then X/Y is a graded N-prime submodule of M/Y.

Also, we characterize graded *N*-prime submodules.

1. Introduction

Dauns introduces the notion of a prime submodule and investigates some of its properties [4]. Graded rings and graded modules have been studied by Nastasescu and Van Oystaeyen [5]. Moreover, based on the definition

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of prime submodules in the sense of Dauns, Atani and Fazalipour have defined the graded prime submodules of graded modules and investigated some properties [1, 2]. The notion of graded primary submodules has been introduced and studied by Oral et al. [6]. Recently, Sanh [7] introduced the prime submodule of fully invariant submodule of R-module M. Let X be a fully invariant proper submodule of M. Then X is called a *prime submodule* of M if for any ideal I of S and any fully invariant submodule U of M, $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$. In this paper, we use the definition of prime submodules in the sense of Sanh and call these *N-prime submodules*. Moreover, we define an N-prime submodule in graded R-modules and we call it a *graded N-prime submodule*. We prove that if X is a fully invariant graded submodule of M, then the residual ideal of X by M is a graded ideal of S. It is also shown that if M is a graded quasi-projective module, X is a graded N-prime submodule of M and $Y \subset X$ is a fully invariant graded submodule of M, then X/Y is a graded N-prime submodule of M/Y. Also, we give the characterization of graded N-prime submodule as stated in Theorem 2.2.

Let G be an abelian group with identity e and R be any ring with unit 1_R. The ring R is called a *graded ring* if $R = \oplus_{g \in G} R_g$, where R_g is an additive subgroup of R and $R_g R_h \subseteq R_{gh}$ for every g, h in G. The summands R_g's are called *homogeneous components*. Also, we write $h(R) = \bigcup_{g \in G} R_g$. If $a \in R$, then a can be written uniquely as $\sum_{g \in G} a_g$, where a_g is a component of a in R_g. In this case, R_e is a subring of R and $1_R \in R_e$.

Let R be a graded ring and M be an R-module. We call M a *graded R-module* if there exists a family of subgroups $\{M_g\}_{g \in G}$ of M such that $M = \oplus_{g \in G} M_g$ and $R_g M_h \subseteq M_{gh}$. The $R_g M_h$ denotes the additive subgroup of M consisting of all finite sum of elements $r_g s_h$, where $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g \in G} M_g$ and the elements of $h(M)$

are called to be *homogeneous*. If M is a graded R-module, then M_g is an R_e-module for all $g \in G$. A submodule X of a graded R-module M is called a *graded submodule* of M if $X = \oplus_{g \in G} X_g$, where $X_g = X \cap M_g$ for $g \in G$. In this case, X_g is called the *g-component* of X. Moreover, M/X becomes a graded module with g-component $(M/X)_{g \in G} = ((M_g + X)/X)_{g \in G}$.

2. Main Results

Let R be a graded ring, M and N be graded R-modules and $f : M \to N$ be an R-module homomorphism. Then f is said to be a *graded R-module homomorphism* of degree k if $f(M_g) \subseteq N_{gk}$ for each $g \in G$, where $k \in G$. Graded homomorphism without an indication of degree is understood to have degree zero. Let $(END_R(M))_k$ be the set of graded module homomorphism from M to M of degree k and let $END_R(M) = \oplus_{k \in G}(END_R(M))_k$. Then $END_R(M)$ is a graded ring and $END_R(M)$ is a subring of $End_R(M)$ (see [3, Subsection 9.1, p. 303]). If G is a finite group, then $END_R(M) = End_R(M)$ (see [5]).

Let M be a graded right R-module and $S = END_R(M)$. A graded submodule X of M is called a *fully invariant graded submodule* of M if for any $s \in S$, $s(X) \subset X$. By the definition, the family of all fully invariant graded submodules of a graded module M is non-empty and closed under intersections and sums.

Let I, J be graded ideals of S and X be a graded submodule of M. We define

$$IJ = \left\{ \sum_{1 \le i \le n} x_i y_i \mid x_i \in h(I),\ y_i \in h(J),\ n \in \mathbb{N} \right\} \text{ and } I(X) = \sum_{f \in h(I)} f(X).$$

For any graded right R-module M and any graded right ideal I of graded ring R, the set MI is a fully invariant graded submodule of M (see [2, Subsection 1]).

Definition 2.1. Let M be a graded right R-module and X be a proper fully invariant graded submodule of M. Then X is called a *graded N-prime submodule* of M if for any graded ideal I of S and any fully invariant graded submodule U of M, $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$.

Especially, if we take M is the R-module R, a graded ideal P of R is a graded prime ideal if for any graded ideals I, J of R with $IJ \subset P$ implies $I \subset P$ or $J \subset P$. From now on, a graded R-module M means a graded right R-module.

The following theorem gives some characterization of graded N-prime submodule.

Theorem 2.2. *Let M be a graded R-module and X be a proper fully invariant graded submodule of M. Then the following are equivalent*:

(1) *X is a graded N-prime submodule of M.*

(2) *For any graded right ideal I of S and graded submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$.*

(3) *For any $\varphi \in h(s)$ and fully invariant graded submodule U of M, if $\varphi(U) \subset X$, then either $\varphi(M) \subset X$ or $U \subset X$.*

Proof. ($1 \Rightarrow 2$) Suppose X is a graded N-prime submodule of M. Take any graded right ideal I of S and a graded submodule U of M where $I(U) \subset X$. Since I is a graded right ideal of S, $IS \subset I$ and SI is a graded ideal of S. Since U is a graded submodule of M, $S(U)$ is a fully invariant graded submodule of M. If $I(U) \subset X$, then $(SI)(S(U)) = (SIS)(U) \subset S(I(U)) \subset S(X) \subset X$. From assumptions that X is a graded N-prime submodule of M, we have $SI(M) \subset X$ or $S(U) \subset X$. Hence, either $I(M) \subset X$ or $U \subset X$.

($2 \Rightarrow 3$) Obvious.

(3⇒1) Take any graded ideal I of S and any fully invariant graded submodule U of M where $I(U) \subset X$. Since I is a graded ideal, I has a set of homogeneous generators. By (3), we obtain $I(M) \subset X$ or $U \subset X$. □

Let M be a graded R-module, $S = END_R(M)$ and X be a fully invariant submodule of M. We define the set $I_X = \{f \in S \mid f(M) \subset X\}$. The set I_X is a graded ideal if X is a fully invariant graded submodule as we give in the following lemma.

Lemma 2.3. *Let M be a graded R-module and $S = END_R(M)$. Suppose that X is a fully invariant graded submodule of M. Then the set I_X is a graded ideal of S.*

Proof. Take any $\varphi \in S$ and $f \in I_X$. It is clear that $(I_X, +)$ is an abelian group. Then $\varphi f(M) \subset \varphi(X) \subset X$ and $f\varphi(M) \subset f(M) \subset X$. So φf, $f\varphi$ in I_X, and we prove that I_X is an ideal of S. Furthermore, we will prove that I_X is a graded ideal of S, i.e., $I_X = \oplus_{g \in G}(I_X \cap S_g)$ for every $g \in G$. For every $g \in G$, $I_X \cap S_g \subset I_X$, so we obtain $\oplus_{g \in G}(I_X \cap S_g) \subset I_X$. Take any $f \in I_X$. Then $f = \sum_{g \in G} f_g$ and $f(M) = \left(\sum_{g \in G} f_g\right)(M) \subset X$. We will prove that $f \in \oplus_{g \in G}(I_X \cap S_g)$. It is clear that $f_g \in S_g$, so we have to prove that $f_g \in I_X$ for every $g \in G$. Without loss of generality, we may assume that $f = \sum_{i=1}^{m} f_{g_i}$, where $f_{g_i} \neq 0$ for all $i = 1, 2, ..., m$ and $f_g = 0$ for all $g \notin \{g_1, g_2, ..., g_m\}$. Since M is a graded module, we assume that $m = \sum_{j=1}^{l} m_{h_j}$, where $m_{h_j} \neq 0$ for all $j = 1, 2, ..., l$. Since $f(M) \subset X$ and $m_{h_j} \in M_{h_j} \subset M$ for all j, we obtain $f(m_{h_j}) \in X$ for all j. Then $\sum_{i=1}^{m} f_{g_i}(m_{h_j}) \in X$, where $f_{g_i}(m_{h_j}) \in M_{g_i h_j}$. Since X is a graded submodule, we obtain $f_{g_i}(m_{h_j}) \in M_{g_i h_j} \cap X \subset X$. Thus, $f_{g_i}(m_{h_j}) \in X$ for all j so $f_{g_i}(M) \subset X$ and $f_{g_i} \in I_X$ for all i, as required. □

It is worth pointing out that I_X is a graded prime if X is a graded N-prime, as we give in the following theorem.

Theorem 2.4. *Let M be a graded R-module,* $S = END_R(M)$ *and X be a fully invariant proper graded submodule of M. If X is a graded N-prime submodule, then* I_X *is a graded prime ideal of S.*

Proof. Let K, L be graded ideals of I_X such that $KL \subset I_X$. Then $KL(M) \subset I_X(M) \subset X$. If we assume that $K \not\subset I_X$, then $K(M) \not\subset X$. Since submodule X is a graded N-prime submodule, $L(M) \subset X$, so we obtain $L \subset I_X$. Thus, I_X is a graded prime ideal of S. □

We define the set $I(M) = \sum_{f \in I} f(M)$. If $I(M) \subset X$, then $I \subset I_X$ and the converse is also true as we prove in the following proposition.

Proposition 2.5. *Let M be a graded R-module, X be a fully invariant graded submodule of M and I be a graded ideal of S. Then* $I(M) \subset X$ *if and only if* $I \subset I_X$.

Proof. Take any $f \in I$, $f(M) \subseteq I(M)$. Since $I(M) \subset X$, we have $f(M) \subset X$. So we have $f \in I_X$. Conversely, consider the set $I(M)$. Since $I \subset I_X$, we have $\sum_{f \in I} f(M) \subset \sum_{f \in I_X} f(M) \subset X$. □

We conclude from Proposition 2.5 and Definition 2.1 and obtain the following theorem.

Theorem 2.6. *Let M be a graded R-module and X be a fully invariant proper graded submodule of M. Then X is a graded N-prime submodule if and only if for any graded ideal I of S and any fully invariant graded submodule U of M such that* $I(U) \subset X$ *implies* $I \subset I_X$ *or* $U \subset X$.

Proof. Let X be a graded N-prime submodule. By Definition 2.1, for any graded ideal I of S and any fully invariant graded submodule U of M such that $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$. According to Proposition 2.5, $I(M) \subset X$ is equivalent to $I \subset I_X$. □

Definition and some properties of a graded *N*-prime module are given as follows.

Definition 2.7. A graded *R*-module *M* is called an *N-prime module* if 0 is a graded *N*-prime submodule of *M*.

We can characterize *N*-prime module using the annihilator as the following proposition.

Proposition 2.8. *Let M be a graded R-module and* $S = END_R(M) = \oplus_{k \in G} END_R(M)_k$. *A module M is an N-prime module if and only if* $Ann_S(M) = Ann_S(X)$ *for all nonzero graded submodules X of M.*

Proof. (⇒) Let *M* be a graded *N*-prime module. Then 0 is a graded *N*-prime submodule of *M*. Since *X* is a nonzero graded submodule of *M*, $Ann_S(M) \subseteq Ann_S(X)$. Take any $f \in Ann_S(X)$, hence $f(X) = 0$. Since *X* is a nonzero graded submodule and 0 is a graded *N*-prime submodule of *M*, we have $f(M) = 0$. Equivalently, $f \in Ann_S(M)$. So we obtain $Ann_S(M) \supseteq Ann_S(X)$ and moreover $Ann_S(X) = Ann_S(X)$.

(⇐) Take any graded ideal *I* of *S* and a nonzero fully invariant graded submodule *X* of *M* where $I(X) = 0$. Since $Ann_S(M) = Ann_S(X)$, $I(M) = 0$. So we obtain 0 is a graded *N*-prime submodule of *M*. It is proved that *M* is a graded *N*-prime module. □

Proposition 2.9. *Let M be a graded N-prime R-module. Then* $S = END_R(M) = \oplus_{k \in G} END_R(M)_k$ *is a prime ring.*

Proof. Let *M* be a graded *N*-prime module. Then 0 is a graded *N*-prime submodule of *M*. Based on Theorem 2.4, I_0 is a graded prime ideal of *S*, so *S* is a prime ring. □

The following proposition states the relations between a graded module homomorphism of M and a graded module homomorphism of M/Y.

Proposition 2.10. *Let M be a graded module, Y be a fully invariant graded submodule of M. If $f : M \to M$ is a graded module homomorphism of degree zero, then $\varphi : M/Y \to M/Y$ with $\varphi(m + Y) = f(m) + Y$ is a graded module homomorphism of degree zero.*

Proof. (i) We will show that φ is a mapping. Take any $m_1 + Y$, $m_2 + Y \in M/Y$ with $m_1 + Y = m_2 + Y$, so $m_1 - m_2 \in Y$. Since Y is a fully invariant graded submodule of M, $f(m_1 - m_2) = f(m_1) - f(m_2) \in Y$, it means $f(m_1) + Y = f(m_2) + Y$. In other words, $\varphi(m_1 + Y) = \phi(m_2 + Y)$.

(ii) It is clear that φ is a module homomorphism.

(iii) We show that ϕ is a graded module homomorphism of degree zero. Take any $m_g + K \in (M_g + Y)/Y$ for some $g \in G$, a homogeneous element of degree g in M/Y. We will prove that $\varphi(M_g + Y) \in (M_g + Y)/Y$. Based on definition of φ, $\varphi(M_g + Y) = f(M_g) + Y$. Since f is a graded module homomorphism of degree zero, $f(m_g) \in M_g$. In other words, $\varphi(m_g + Y) \in (M_g + Y)/Y$. It is proved that $\varphi((M_g + Y)/Y) \subseteq (M_g + Y)/Y$ or φ is a graded module homomorphism of degree zero. □

We will look more closely at the properties of graded N-prime submodule of quotient module.

Lemma 2.11. *Let M be a graded module, X, Y be graded submodules of M and $Y \subset X$. Then X/Y is a graded submodule of M/Y.*

Proof. It is clear that X/Y is a submodule of M/Y. Furthermore, we will show that X/Y is a graded submodule. It means, we will show that $X/Y = \oplus_{g \in G} X/Y \cap (M/Y)_g$. The condition $\oplus_{g \in G} X/Y \cap (M/Y)_g \subseteq X/Y$

is obvious. Let $\overline{m} = \sum_{g \in G} \overline{m}_g \in X/Y$. It is sufficient to show that $\overline{m}_g = m_g + Y \in X/Y \cap (X/Y)_g$ for each $g \in G$. Since X is a graded submodule of M, $m_g \in X \cap M_g$ for each $g \in G$, so $m_g \in X$ and $m_g \in M_g$. Then $m_g + Y \in X/Y$ and $m_g + Y \in (M_g + Y)/Y = (M/Y)_g$. Hence, $\overline{m}_g = m_g + Y \in X/Y \cap (M/Y)_g$. □

Theorem 2.12. *Let M be a graded quasi-projective module, X be a graded N-prime submodule of M and* $Y \subset X$ *be a fully invariant graded submodule of M. Then* X/Y *is a graded N-prime submodule of* M/Y.

Proof. Let $\overline{S} = END_R(M/Y)$. Let φ be a homogeneous element of degree zero in $\overline{S}$ and U/Y be a fully invariant graded submodule of M/Y with $Y \subset U$ and $\varphi(U/Y) \subset X/Y$. Since M is quasi-projective, we can find $f \in h(S) = h(END_R(M))$, f is a homogeneous element of degree zero in S such that $\varphi v = vf$, where $v : M \to M/Y$ is the graded canonical projection. Then $\varphi(U/Y) = \varphi v(U) = vf(U) = (f(U) + Y)/Y \subset X/Y$. It follows that $f(U) \subset X$. Since Y is a fully invariant graded submodule of M and U/Y is a fully invariant graded submodule of M/Y, U is a fully invariant graded submodule of M. By the primeness of X, we have $f(M) \subset X$ or $U \subset X$. Thus, $(f(M) + Y)/Y = vf(M) = \varphi v(M) = \varphi(M/Y) \subset X/Y$ or $U/Y \subset X/Y$, that is, X/Y is a graded N-prime submodule of M/Y. □

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